Cosmography of $f(R)$ - brane cosmology

Mariam Bouhmadi-López[1*], Salvatore Capozziello[2,3†], Vincenzo F. Cardone[2,4‡]
[1] *Centro Multidisciplinar de Astrofísica - CENTRA, Departamento de Física, Instituto Superior Técnico, Av. Rovisco Pais 1, 1049-001 Lisboa, Portugal*
[2] *Dipartimento di Scienze Fisiche, Università di Napoli "Federico II", Compl. Univ. Monte S. Angelo, Ed.N, Via Cinthia, I-80126 Napoli, Italy*
[3] *I.N.F.N. - Sez. di Napoli, Compl. Univ. Monte S. Angelo, Ed.G, Via Cinthia, I-80126 Napoli, Italy and*
[4] *Dipartimento di Scienze e Tecnologie dell' Ambiente e del Territorio, Università degli Studi del Molise, Contrada Fonte Lappone, 86090 - Pesche (IS), Italy*

Cosmography is a useful tool to constrain cosmological models, in particular dark energy models. In the case of modified theories of gravity, where the equations of motion are generally quite complicated, cosmography can contribute to select realistic models without imposing arbitrary choices *a priori*. Indeed, its reliability is based on the assumptions that the universe is homogeneous and isotropic on large scale and luminosity distance can be "tracked" by the derivative series of the scale factor $a(t)$. We apply this approach to induced gravity brane-world models where an $f(R)$-term is present in the brane effective action. The virtue of the model is to self-accelerate the normal and healthy DGP branch once the $f(R)$-term deviates from the Hilbert-Einstein action. We show that the model, coming from a fundamental theory, is consistent with the ΛCDM scenario at low redshift. We finally estimate the cosmographic parameters fitting the Union2 Type Ia Supernovae (SNeIa) dataset and the distance priors from Baryon Acoustic Oscillations (BAO) and then provide constraints on the present day values of $f(R)$ and its second and third derivatives.

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I. INTRODUCTION

The late-time acceleration of the Universe has been confirmed by several observations ranging from type Ia Supernovae (SNeIa) [1], which brought the first evidence, to the cosmic microwave background (CMB) [2] and the baryon acoustic oscillations (BAO) [3]. More recently, gamma ray bursts (GRB), also if not properly standard candles, have been as well very useful at this regard [4, 5]. They could, in principle, be useful to probe high redshifts with the aim to remove degeneracy of cosmological models with respect to ΛCDM [6, 7]. While the recent speed up of the universe is a fact, we have yet no answer to the question: What is the "hand that rocks the cradle"?

If we assume that general relativity is valid on all the scales, even though it has been corroborated at most on the solar system range, then we require a component on the budget of the universe, that violates at least the strong energy condition to describe the current acceleration of the universe [8]. The simplest option at this regard corresponds to a cosmological constant, giving raise to the ΛCDM model which matches pretty well the observations, but then we face the cosmological constant problem. An alternative approach is to invoke a gravitational theory that deviates from general relativity on the appropriate scales and at the same time being able to reproduce the big achievements of general relativity (cf. Refs. [9–13]). The latter approach can be tackled in the context of brane-world models [14], which are inspired in string theory, where our universe corresponds to a 4-dimensional hypersurface embedded on the higher dimensional spacetime, usually dubbed the bulk. Several approach have been undertaken, for example in the context of induced gravity brane-world [16, 17] the self-accelerating brane of the Dvali-Gabadadze-Porrati (DGP) model is probably the most famous [15].

The DGP model has gathered a lot of attention on the last years. As an induced gravity brane-world model, it contains two possible solutions, the self-accelerating branch, which is asymptotically de Sitter, and the normal branch. Despite this fact, the self-accelerating brane does not require any type of dark energy to describe a late-time inflationary period of the brane, it suffers from some theoretical problems like the ghost problem [18]; i.e. a degree of freedom that shows up when the brane is perturbed and behaves on the brane effectively as a scalar field with the wrong kinetic energy. On the other hand, the normal branch is "healthy" in the sense that it does not suffer from the ghost problem but it requires some sort of dark energy to describe the late-time acceleration of the universe.

In a previous paper [19], one of us proposed a mechanism to self-accelerate the normal DGP branch. More precisely, a generalized induced gravity brane-world model is proposed where the brane action contains an arbitrary $f(R)$ term, R being the scalar curvature of the brane[1]. It is shown that an $f(R)$ ($\neq R$) term on the dy-

*mariam.bouhmadi@ist.utl.pt
†capozziello@na.infn.it
‡winnyenodrac@gmail.com

[1] See Ref. [20] for a brane-world model with an $f(R)$ term in the

namics of a homogeneous and isotropic brane induces a shift on the energy density of the brane. This new shift term, which is absent in the DGP model, plays a crucial role to self-accelerate the generalized normal DGP branch of the model. In other terms, the generalized normal branch is asymptotically de Sitter without considering any dark energy on the brane.

In the present paper, we discuss the possibility to constrain this model using a cosmographic approach [21]. Cosmography relies on two crucial things: i) extracting the maximum amount of information from measured distances, like the luminosity distances of SNeIa, ii) assuming that the universe can be modelled by a Friedmann-Lemaître-Robertson-Walker (FLRW) model on large scale without assuming *a priori* any dynamical theory to describe it. Now, why have we chosen this approach? for several reasons: i) for its simplicity. For example, the modified Einstein equation of the brane are of fourth order on the scale factor (due to the $f(R)$-term in the brane action) and therefore very difficult to solve analytically. In the cosmographic approach we do not need to have an explicit solution for the evolution of the scale factor in terms of the cosmic time of the brane. ii) The approach is quite general in the sense that we do not have to specify which $f(R)$ function we are dealing with. The only requirement is that $f(R)$ is an analytic function.

The outline of the paper is as follows. In Sect.II, we review the model presented in [19]. In particular, we highlight how the model contains fixed points corresponding to de Sitter solutions (in absence of any matter on the brane); i.e. self-accelerating solutions. In Sect.III, we present the cosmographic approach we will follow. We write down all the quantities relevant of the model in terms of the cosmographic parameters. In Sect.IV, we constrain the model from a theoretical point of view, while Sect.V deals with observational constraints. Finally, we summarize and discuss the obtained results in Sect.VI.

II. AN $f(R)$-TERM ON THE BRANE

In this section, we review the model introduced in [19]. The scenario corresponds to a 5-dimensional brane-world model whose action reads

$$\begin{aligned} \mathcal{S} &= \int_{\mathcal{B}} d^5X\,\sqrt{-g^{(5)}}\,\left\{\frac{1}{2\kappa_5^2}R[g^{(5)}]\right\} \\ &\quad + \int_h d^4X\,\sqrt{-g}\,\left\{\frac{1}{\kappa_5^2}K \;+\; \frac{1}{2\kappa_4^2}f(R)+\mathcal{L}_m\right\}, \end{aligned} \tag{2.1}$$

where κ_5^2 is the 5D gravitational constant, $R[g^{(5)}]$ is the scalar curvature in the bulk and K the extrinsic curvature of the brane in the higher dimensional bulk. For

bulk.

the sake of simplicity, we have assumed a vanishing bulk cosmological constant, for a more general setup please see [19]. In addition, R is the scalar curvature of the induced metric on the brane, g, and κ_4^2 is related to the Newtonian gravitational constant, G, through $\kappa_4^2 = 8\pi G$. The function $f(R)$ has mass square units. On the other hand, $\mathcal{L}_m$ corresponds to the standard matter Lagrangian of the brane. We recover the DGP model [15, 16] when $f(R) = R$.

From now on, we assume a homogeneous and isotropic brane with spatially flat sections. Therefore, the modified Friedmann equation can be written as

$$3H^2 = \frac{\kappa_5^4}{12}\rho^2. \tag{2.2}$$

The total energy density ρ is conserved and is given by

$$\rho = \rho_m + \rho_f, \tag{2.3}$$

where

$$\begin{aligned} \rho_m &= \frac{\rho_{m0}}{a^3}, \\ \rho_f &= -\frac{1}{\kappa_4^2}\left[3H^2f' - \frac{1}{2}(Rf'-f) + 3H\dot{R}f''\right], \end{aligned} \tag{2.4}$$

where both energy densities ρ_m and ρ_f are conserved separately. We will use the subscript 0 to refer to quantities evaluated at the present time. The dot stands for derivative with respect to the cosmic time of the brane and the prime for derivative respect to the scalar curvature of the brane.

We are interested on the branch that generalize the standard DGP solution and therefore the modified Friedmann equation (2.2) reduces to

$$H = \frac{\kappa_5^2}{6}\rho. \tag{2.5}$$

The other root of Eq. (2.2) generalizes the Friedmann equation of the self-accelerating DGP solution.

For latter convenience it is useful to rewrite Eq. (2.5) as

$$f'H^2 + \frac{1}{r_c}H = \frac{\kappa_4^2}{3}\frac{\rho_{m0}}{a^3} + \frac{1}{6}\left(Rf' - f - 6H\dot{R}f''\right). \tag{2.6}$$

The parameter $r_c = \kappa_5^2/(2\kappa_4^2)$ is the crossover scale. For $1 \ll f'r_cH$, we obtain the Friedmann equation for 4-dimensional $f(R)$ models.

The Raychaudhuri equation for this model can be deduced by taking the time derivative of Eq. (2.6), bearing in mind that the matter energy density is conserved, and it reads

$$\dot{H} + \frac{1}{r_c}\frac{1}{2f'}\frac{\dot{H}}{H} = -\frac{\kappa_4^2}{2f'}\frac{\rho_{m0}}{a^3} - \frac{\dot{R}^2f''' + (\ddot{R} - H\dot{R})f''}{2f'}. \tag{2.7}$$

To obtain this equation we have as well used[2] $R = 6(2H^2 + \dot{H})$.

It can be shown that the brane contains fixed points corresponding to de Sitter solutions (once the matter content is negligible) [19], therefore the brane enters a self-accelerating regime at some point along its expansion. In reference [19], it is shown what are the conditions to be fulfilled for the de Sitter solutions to be stable under homogeneous perturbations [19]. More precisely, we can associate an effective square mass to the perturbations and, as long as this quantity is positive, we can conclude that de Sitter solution is stable.

III. COSMOGRAPHY

A. General approach

As we said, cosmography relies on the assumption that the universe is homogeneous and isotropic on large scale and no dynamical theory is assumed a priori [21]. In particular it relies on the scale factor series expansion of a FLRW metric in terms of time [21]; i.e.

$$\begin{aligned} \frac{a(t)}{a(t_0)} &= 1 + H_0(t-t_0) - \frac{q_0}{2}H_0^2(t-t_0)^2 \\ &+\frac{j_0}{3!}H_0^3(t-t_0)^3 + \frac{s_0}{4!}H_0^4(t-t_0)^4 \\ &+\frac{l_0}{5!}H_0^5(t-t_0)^5 + O((t-t_0)^6) \end{aligned} \tag{3.1}$$

where the standard cosmographic parameters are defined as [21]

$$\begin{aligned} H &= \frac{1}{a}\frac{da}{dt} \\ q &= -\frac{1}{a}\frac{d^2a}{dt^2}\ H^{-2} \\ j &= \frac{1}{a}\frac{d^3a}{dt^3}\ H^{-3} \\ s &= \frac{1}{a}\frac{d^4a}{dt^4}\ H^{-4} \\ l &= \frac{1}{a}\frac{d^5a}{dt^5}\ H^{-5}. \end{aligned} \tag{3.2}$$

These parameters are usually referred to as the Hubble, deceleration, jerk, snap and lerk parameters respectively (see [21] and references therein). Their present day values (which we will denote with a subscript 0) can be used to characterize the evolutionary status of the Universe. For example, $q_0 < 0$ denotes an accelerated expansion, while a change of sign of j (in an expanding universe) signals that the acceleration starts increasing or decreasing.

Most importantly, the parameters $\{q_0, j_0, l_0, s_0\}$ can be used to evaluate different distances in the universe. This can be achieved by inverting the relation (3.1) and bearing in mind that the distance, D, travelled by a given photon that was emitted at t_1 and detected at the current epoch t_0 is simply $D = t_0 - t_1$ (where we have set the speed of light to unity). Therefore, one can obtain a series expansion of the distance D in terms of the scale factor or redshisft, while the coefficients of the expansion are defined through the cosmographic parameters [21]. The distance D can be related to several physical magnitude, for example the luminosity distance, the angular diameter distance and many more [22]. These magnitudes can be constrained observationally through SNeIa, BAO and, possibly, GRB data [4]. In fact, these data are useful to construct a cosmic ladder where any step is a cosmic indicator. Once the distances are constrained, we obtain as well constraints on the values acquired by the cosmographic parameter (see for example [21–23]). It is worthy to notice, at this regard, that given that the cosmographic approach is based on a Taylor expansion of the scale factor, or redshift, for data of GRB at high redshift (above $z = 1$), it is better to use the variable $y = z/(1+z)$, introduced in [24], instead of the redshift.

B. Applying cosmography to $f(R)$ brane-world

In this subsection, we will relate the characteristic quantities defining the model introduced in Sect.II to the parameters $\{q_0, j_0, l_0, s_0\}$. In addition, this will be done without specifying a particular $f(R)$ model on the brane.

We start reminding that the derivative of the Hubble parameter can be expressed in terms of the cosmographic parameters. Indeed, after some algebra, the following relation can be obtained:

$$\dot{H} = -H^2(1+q)\ , \tag{3.3}$$

$$\ddot{H} = H^3(j + 3q + 2)\ , \tag{3.4}$$

$$\dddot{H} = H^4\left[s - 4j - 3q(q+4) - 6\right]\ , \tag{3.5}$$

$$d^4H/dt^4 = H^5\left[l - 5s + 10(q+2)j + 30(q+2)q + 24\right]\ . \tag{3.6}$$

Now, the question is how our model can be characterized by these parameters, or, more precisely, what can be said about the current values of $f(R_0), f'(R_0), f''(R_0), f'''(R_0)$. In order to answer this question we have first to rewrite $R, \dot{R}, \ddot{R}, \dddot{R}$ in terms of q, j, s, l. This can be done with some algebra

[2] We use Wald's book sign convention.

$$\dot{R} = 6\left(\ddot{H} + 4H\dot{H}\right),$$

$$\ddot{R} = 6\left(\dddot{H} + 4H\ddot{H} + 4\dot{H}^2\right), \qquad . \tag{3.7}$$

$$\dddot{R} = 6\left(d^4H/dt^4 + 4H\dddot{H} + 12\dot{H}\ddot{H}\right).$$

Now using Eqs. (3.3), (3.4), (3.5) and (3.6), we get

$$R = 6H^2(1-q)\ , \tag{3.8}$$

$$\dot{R} = 6H^3(j-q-2)\ , \tag{3.9}$$

$$\ddot{R} = 6H^4\left(s + q^2 + 8q + 6\right)\ , \tag{3.10}$$

$$\dddot{R} = 6H^5\left[l - s - 2(q+4)j - 6(3q+8)q - 24\right]\ . \tag{3.11}$$

If we substitute Eqs, (3.3), (3.4), (3.5), (3.6), (3.8), (3.9), (3.10) and (3.11) in the Friedmann and Raychaudhuri equations, and evaluate them at the present time, we could obtain, in principle, the current values of $f(R_0), f'(R_0), f''(R_0), f'''(R_0)$. However as we have only two equations, the Friedmann relation and the Raychaudhuri equation, we require more information to define completely the model. At this respect, notice that the effective gravitational constant on the brane $G_{\rm eff} = G/f'$ (see the Friedmann equation (2.6)), therefore we can assume, as a prior, that $f'(R_0) = 1$ such that the current value of the gravitational constant coincides with the Newtonian one. Further information can be obtained through the equation satisfied by $\dddot{H}$. At this respect, we take the time derivative of Eq. (2.7) and we obtain

$$\begin{aligned}\ddot{H} \ &+\ \frac{1}{2r_c}\frac{(\ddot{H}H - \dot{H}^2)f' - H\dot{H}f''\dot{R}}{(Hf')^2} = \\ &\frac{\dot{R}^2 f''' + (\ddot{R} - \dot{R}H)f'' + \kappa_4^2\rho_{m0}a^{-3}}{2f'^2(f''\dot{R})^{-1}} \\ &-\ \frac{\dot{R}^3 f^{(iv)} + (3\ddot{R}\dot{R} - H\dot{R}^2)f'''}{2f'} \\ &-\ \frac{(\dddot{R} - \ddot{R}H - \dot{R}\dot{H})f'' - 3\kappa_4^2 H\rho_{m0}a^{-3}}{2f'},\end{aligned} \tag{3.12}$$

where $f^{(iv)} = d^4f/dR^4$. As third assumption, we take into account the power series

$$\begin{aligned}f(R) \ &\simeq\ f(R_0) + f'(R_0)(R-R_0) + \frac{1}{2}f''(R_0)(R-R_0)^2 \\ &+\frac{1}{6}f'''(R_0)(R-R_0)^3.\end{aligned} \tag{3.13}$$

i.e. at low redshift, the $f(R)$-function is well approximated by its Taylor expansion up to the third order[3].

Now we can finally substitute Eqs. (3.3), (3.4), (3.5), (3.8), (3.9), (3.10) and (3.11) in the Friedmann constraint (2.6), the Raychaudhuri relation (2.7) and the complementary equation (3.12). We evaluate them at the present time. Notice that Eqs. (2.6) and (2.7) can be expressed as linear combinations of $f(R_0), f'(R_0), f''(R_0), f'''(R_0)$ at $z=0$. This is not the case for equation (3.12) as it is quadratic on $f''(R_0)$. So, we will proceed as follows, we obtain $f(R_0)$ as a linear combination of $f''(R_0)$ using Eq. (2.6),

$$f(R_0) = 6\left[\left(\Omega_m - 1\right)H_0^2 + \frac{1}{6}\left(R_0 - 6H_0\dot{R}_0 f''\right) - \frac{1}{r_c}H_0\right], \tag{3.14}$$

where $\Omega_0 = \kappa_4^2\rho_{m_0}/(3H_0^2)$. Similarly, we can write $f'''(R_0)$ as a linear combination of $f''(R_0)$ using Eq. (2.7), i.e.

$$f'''(R_0) = -\frac{3H_0^2\Omega_m + \dot{H}_0(2 + \frac{1}{r_cH_0}) + (\ddot{R}_0 - H\dot{R}_0)f''(R_0)}{\dot{R}_0^2}. \tag{3.15}$$

Then we rewrite Eq.(3.12) as follows

$$a_2 f''(R_0)^2 + a_1 f''(R_0) + a_0 = 0, \tag{3.16}$$

where

$$\begin{aligned} a_2 \ &=\ \dot{R}_0(\ddot{R}_0 - \dot{R}_0H_0), & (3.17)\\ a_1 \ &=\ \dot{R}_0^3 f'''(R_0) + 3H_0^2\Omega_m\dot{R}_0 - \left(\dddot{R}_0 - \ddot{R}_0 - \dot{R}_0\dot{H}_0\right) \\ &\quad + \frac{\dot{H}_0\dot{R}_0}{r_cH_0} & (3.18)\\ a_0 \ &=\ -\left(3\ddot{R}_0\dot{R}_0 - H_0\dot{R}_0^2\right)f'''(R_0) + 9H_0^3\Omega_m - 2\ddot{H}_0 \\ &\quad - \frac{\ddot{H}_0H_0 - \dot{H}_0^2}{r_cH_0^2}. & (3.19)\end{aligned}$$

Even though the previous equation looks quadratic in $f''(R_0)$, it is not the case because a_1 is a linear function of $f'''(R_0)$ and therefore this term contributes quadratically in $f''(R_0)$. Once we substitute Eq. (3.15) on Eq. (3.16), we obtain a linear equation for $f''(R_0)$.

Finally, we obtain the following results:

$$\begin{aligned}\frac{f(R_0)}{6H_0^2} \ &=\ -\frac{\mathcal{A}_0\Omega_m + \mathcal{B}_0 + \mathcal{C}_0(r_cH_0)^{-1}}{\mathcal{D}}, & (3.20)\\ \frac{f''(R_0)}{(6H_0^2)^{-1}} \ &=\ -\frac{\mathcal{A}_2\Omega_m + \mathcal{B}_2 + \mathcal{C}_2(r_cH_0)^{-1}}{\mathcal{D}}, & (3.21)\\ \frac{f'''(R_0)}{(6H_0^2)^{-2}} \ &=\ -\frac{\mathcal{A}_3\Omega_m + \mathcal{B}_3 + \mathcal{C}_3(r_cH_0)^{-1}}{(j_0 - q_0 - 2)\mathcal{D}}, & (3.22)\end{aligned}$$

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[3] In what follows we assume $f'(R_0) = 1$ and $f^{(iv)}(R_0) \simeq 0$.

where $\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}_i$ and $\mathcal{D}$ with $i = 0, 2, 3$ are functions of q, j, s, l which are defined as

$$
\begin{aligned}
\mathcal{A}_0 &= (j_0 - q_0 - 2)\, l_0 \\
&- \left(3s_0 + 7j_0 + 6q_0^2 + 41q_0 + 22\right) s_0 \\
&- \left[(3q_0 + 16)\, j_0 + 20q_0^2 + 64q_0 + 12\right] j_0 \\
&- \left(3q_0^4 + 25q_0^3 + 96q_0^2 + 72q_0 + 20\right), \qquad (3.23)\\
\mathcal{B}_0 &= -\left(j_0 q_0 - q_0^2 - 2q_0\right) l_0 \\
&+ \left[3q_0 s_0 + (4q_0 + 6)\, j_0 + 6q_0^3 + 44q_0^2 + 22q_0 \right.\\
&\left. -12\right] s_0 \\
&+ \left[2j_0^2 + \left(3q_0^2 + 10q_0 - 6\right) j_0 + 17q_0^3 + 52q_0^2 \right.\\
&\left. +54q_0 + 36\right] j_0 \\
&+ 3q_0^5 + 28q_0^4 + 118q_0^3 + 72q_0^2 - 76q_0 - 64, \qquad (3.24)\\
\mathcal{C}_0 &= -(j_0 - q_0 - 2)\, l_0 \\
&+ \left[3s_0 + (3q_0 + 1)\, j_0 + 3\, q_0^2 + 41q_0 + 34\right] s_0 \\
&+ \left[j_0^2 - \left(q_0^2 - q_0 - 6\right) j_0 + 5q_0^3 + 43q_0^2 \right.\\
&\left. +50q_0 + 4\right] j_0 \\
&- (q_0^4 + 3q_0^3 - 80q_0^2 - 144q_0 - 68), \qquad (3.25)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}_2 &= 9s_0 + 6j_0 + 9q_0^2 + 66q_0 + 42, \qquad (3.26)\\
\mathcal{B}_2 &= -\left\{6\,(q_0 + 1)\, s_0 + 2\left[j_0 + (q_0 - 1)\right] j_0 \right.\\
&\left. + 6q_0^3 + 50q_0^2 + 74q_0 + 32\right\}, \qquad (3.27)\\
\mathcal{C}_2 &= -\left\{3\,(1 + q_0)\, s_0 + \left[j_0 - (q_0^2 + q_0 + 2)\right] j_0 \right.\\
&\left. + 4q_0^3 + 29q_0^2 + 42q_0 + 18\right\}, \qquad (3.28)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}_3 &= -3\left[l_0 + s_0 - 3\,(q_0 + 4)\, j_0 - 15q_0^2 \right.\\
&\left. -26q_0 - 4\right], \qquad (3.29)\\
\mathcal{B}_3 &= 2\left[(1 + q_0)\, l_0 + (q_0 + j_0)\, s_0 \right.\\
&- \left(j_0 + 2q_0^2 + 6q_0 + 3\right) j_0 \\
&\left. - \left(15q_0^3 + 42q_0^2 + 39q_0 + 12\right)\right], \qquad (3.30)\\
\mathcal{C}_3 &= (1 + q_0)\, l_0 + \left(j_0 - q_0^2 - q_0 - 1\right) s_0 \\
&- \left(j_0 + q_0^2 + 4q_0 + 2\right) j_0 \\
&- (q_0^4 + 26q_0^3 + 69q_0^2 + 64q_0 + 20), \qquad (3.31)\\
\mathcal{D} &= -(j_0 - q_0 - 2)\, l + \left(3s_0 - 2j_0 + 6q_0^2 + 50q_0 + 40\right) s_0 \\
&+ \left[(3q_0 + 10)\, j + 11q_0^2 + 4q_0 - 18\right] j_0 \\
&+ 3q_0^4 + 34q_0^3 + 180q_0^2 + 246q_0 + 104. \qquad (3.32)
\end{aligned}
$$

We have split the expressions of $f(R_0)$, $f''(R_0)$ and $f'''(R_0)$ into three pieces involving the functions $\mathcal{A}_i\Omega_m$, $\mathcal{B}_i$ and $\mathcal{C}_i(r_cH_0)^{-1}$, where $\mathcal{A}_i$, $\mathcal{B}_i$ and $\mathcal{C}_i$ are defined exclusively in terms of the cosmographic parameters. The first term $\mathcal{A}_i\Omega_m$ account for the contribution of matter to the $f(R)$-function[4]. The second one $\mathcal{B}_i$ is a purely geometrical one. The third one takes into account the effect of the extra dimension; i.e. it involves the crossover scale r_c. Not surprisingly, if we switch off this term; i.e. $1 \ll r_c$, we recover exactly the results obtained in [21] corresponding to a standard 4-dimensional $f(R)$ scenario.

In summary, for a given set of values of the cosmographic parameters we can deduce the function $f(R)$ through the expression (3.13). Notice that the opposite is not possible because the equations (3.23)-(3.32) are non-linear in $\{q_0, s_0, l_0, j_0\}$. Moreover, by specifying a given function $f(R)$, we do not obtain a unique evolution for the brane because the modified Raychaudhuri equation is of fourth order in the scale factor.

[4] It is worth noticing that we are developing our considerations in the Jordan frame so the standard matter is minimally coupled to the geometry.

IV. PARAMETERIZING THE COSMOGRAPHIC PARAMETERS

In order to get a first hint on the possible values of $f(R)$ and its derivatives we adopt the following strategy: the cosmographic parameters will be calculated for a given dark energy phenomenological parameterization. The best and simplest one is the ΛCDM model. Next, we will evaluate those parameters using the recent data of WMAP7 and the constraint on the crossover scale r_c (see [25] for details). Through these results, we can constrain the $f(R)$ function as we will show below. This is a minimal approach but it is useful to probe the self-consistency of the model.

The cosmographic parameters for the ΛCDM model read

$$
\begin{aligned}
q &= -\left(\frac{H_0}{H}\right)^2 \left(1 - \Omega_m - \frac{1}{2}\frac{\Omega_m}{a^3}\right), \qquad (4.1)\\
j &= \left(\frac{H_0}{H}\right)^3 \left(1 - \Omega_m + \frac{\Omega_m}{a^3}\right)^{\frac{3}{2}}, \qquad (4.2)\\
s &= \left(\frac{H_0}{H}\right)^4 \left(1 - 2\Omega_m - \frac{5}{2}\frac{\Omega_m}{a^3} + \Omega_m^2 \right.\\
&\left. + \frac{5}{2}\frac{\Omega_m^2}{a^3} - \frac{7}{2}\frac{\Omega_m^2}{a^6}\right), \qquad (4.3)\\
l &= \left(\frac{H_0}{H}\right)^5 \left(1 - 2\Omega_m + 5\frac{\Omega_m}{a^3} + \Omega_m^2 \right.\\
&\left. -5\frac{\Omega_m^2}{a^3} + \frac{35}{2}\frac{\Omega_m^2}{a^6}\right)\\
&\times\sqrt{1 - \Omega_m + \frac{\Omega_m}{a^3}}, \qquad (4.4)
\end{aligned}
$$

which, evaluated at the present time, give [21]

$$
\begin{aligned}
q_0 &= -1 + \frac{3}{2}\Omega_m, \qquad (4.5)\\
j_0 &= 1, \qquad (4.6)\\
s_0 &= 1 - \frac{9}{2}\Omega_m, \qquad (4.7)\\
l_0 &= 1 + 3\Omega_m + \frac{27}{2}\Omega_m^2. \qquad (4.8)
\end{aligned}
$$

Inserting the previous equations in the equations (3.22)-(3.31), we obtain

$$\mathcal{A}_0 = -\frac{63}{4}\Omega_m^2 - \frac{27}{8}\Omega_m^3 - \frac{243}{16}\Omega_m^4, \tag{4.9}$$

$$\mathcal{B}_0 = -63\Omega_m^2 + \frac{27}{2}\Omega_m^3 + \frac{81}{16}\Omega_m^4 + \frac{729}{32}\Omega_m^5, \tag{4.10}$$

$$\mathcal{C}_0 = 63\Omega_m^2 + \frac{81}{8}\Omega_m^3 - \frac{81}{16}\Omega_m^4, \tag{4.11}$$

$$\mathcal{A}_2 = \frac{63}{2}\Omega_m + \frac{81}{4}\Omega_m^2, \tag{4.12}$$

$$\mathcal{B}_2 = -\frac{63}{2}\Omega_m^2 - \frac{81}{4}\Omega_m^3, \tag{4.13}$$

$$\mathcal{C}_2 = 189\Omega_m - \frac{441}{4}\Omega_m^2 - 84 - \frac{27}{2}\Omega_m^3, \tag{4.14}$$

$$\mathcal{A}_3 = \frac{243}{4}\Omega_m^2, \tag{4.15}$$

$$\mathcal{B}_3 = -\frac{243}{4}\Omega_m^3, \tag{4.16}$$

$$\mathcal{C}_3 = -\frac{351}{8}\Omega_m^3 - \frac{81}{16}\Omega_m^4, \tag{4.17}$$

$$\mathcal{D} = 63\Omega_m^2 + \frac{135}{4}\Omega_m^3 + \frac{243}{16}\Omega_m^4. \tag{4.18}$$

It can be checked that if $1 \ll r_c$; i.e. in absence of an extra dimension, the function $f(R)$ reduces to $f(R) \sim R - 2\Lambda$ because $f''(R_0) = 0$ and $f'''(R_0) = 0$. This can be assumed as a consistency check. However, as soon as the effect of the extra dimension is switched on, i.e. r_c is finite, the coefficients $\mathcal{C}_i$ with $i = 0, 2, 3$ play a crucial in defining the shape of the function $f(R)$. Indeed, we obtain

$$F_{GR_0} \equiv -\frac{\mathcal{A}_0\Omega_m + \mathcal{B}_0}{\mathcal{D}} \tag{4.19}$$

$$= -\frac{1}{2}\Omega_m + 1 = \frac{R_0 - 2\Lambda}{6H_0^2}, \tag{4.20}$$

$$F_{GR_2} \equiv -\frac{\mathcal{A}_2\Omega_m + \mathcal{B}_2}{\mathcal{D}} = 0, \tag{4.21}$$

$$F_{GR_3} \equiv -\frac{\mathcal{A}_3\Omega_m + \mathcal{B}_3}{(j_0 - q_0 - 2)\mathcal{D}} = 0, \tag{4.22}$$

$$F_{IG_0} \equiv -\frac{\mathcal{C}_0(r_cH_0)^{-1}}{\mathcal{D}} \tag{4.23}$$

$$\equiv \frac{-112 - 18\Omega_m + 9\Omega_m^2}{112 + 60\Omega_m + 27\Omega_m^2}(r_cH_0)^{-1}, \tag{4.24}$$

$$F_{IG_2} \equiv -\frac{\mathcal{C}_2(r_cH_0)^{-1}}{\mathcal{D}} \tag{4.25}$$

$$= \frac{4}{3}\frac{-252\Omega_m + 147\Omega_m^2 + 112 + 18\Omega_m^3}{\Omega_m^2(112 + 60\Omega_m + 27\Omega_m^2)}(r_cH_0)^{-1}, \tag{4.26}$$

$$F_{IG_3} \equiv -\frac{\mathcal{C}_3(r_cH_0)^{-1}}{(j_0 - q_0 - 2)\mathcal{D}} \tag{4.27}$$

$$= 3\Omega_m\frac{26 + 3\Omega_m}{112 + 60\Omega_m + 27\Omega_m^2}(r_cH_0)^{-1}. \tag{4.28}$$

For clarity, we have split the right hand side (rhs) of Eq. (3.20) into two pieces: F_{GR_0} and F_{IG_0}, the first one takes into account the pure relativistic contribution while the second one takes into account the effect of the extra dimension. A similar procedure has been followed with the rhs of Eqs. (3.21) and (3.22).

We consider the following observational conservative values $\Omega_m = 0.266$ and $\Omega_{r_c} = 10^{-4}$ where $\Omega_{r_c} = (4r_cH_0^2)^{-1}$ [2, 25] and we obtain the values reported below:

$$F_{GR_0} = 0.867\,,$$
$$F_{GR_2} = 0\,,$$
$$F_{GR_3} = 0\,,$$
$$F_{IG_0} = -0.018\,,$$
$$F_{IG_2} = 0.161\,,$$
$$F_{IG_3} = 0.003\,,$$

with the errors evaluated as in [2, 25]. The previous results show that, although F_{IG_i} are different from zero, they are relatively small in comparison with the present day main contribution F_{GR_0}; i.e. the standard relativistic term. In summary, the model deviates just slightly from the pure ΛDGP model[5] [26, 27]. This small deviation is enough to obtain self-acceleration without invoking any kind of dark energy contribution on the brane. On the other hand, if a similar analysis is carried out for a given $f(R)$ function in a 4-dimensional model, it turns out that the $f(R)$-term match completely that of a Hilbert-Einstein action plus a cosmological constant [21]. Most importantly, we see that the model we have analyzed is consistent with the ΛCDM model because the cosmographic parameters of the ΛCDM can be matched to those of an $f(R)$ brane-world scenario.

V. OBSERVATIONAL CONSTRAINTS

In order to constrain the model, i.e. to estimate the function $f(R)$ through its own value and that of its derivatives at the present time, we need to constrain observationally the cosmographic parameters by using appropriate distance indicators. Moreover, we must take care that the expansion of the distance related quantities in terms of (q_0, j_0, s_0, l_0) closely follows the exact expressions over the range probed by the data used. Taking SNeIa and a fiducial ΛCDM model as a test case, one has to check that the approximated luminosity distance[6] deviates from the ΛCDM one less than the measurement uncertainties up to $z \simeq 1.5$ to avoid introducing any systematic bias. Since we are interested in constraining (q_0, j_0, s_0, l_0), we will expand the luminosity distance D_L up to the fifth order in z which indeed allows us

[5] The ΛDGP model corresponds to the normal DGP branch endowed with a cosmological constant and filled with matter.

[6] See [21] for the analytical expression.

x	x_{BF}	$\langle x\rangle$	x_{med}	68% CL	95% CL
h	0.744	0.750	0.750	(0.725, 0.775)	(0.701, 0.802)
q_0	-0.43	-0.44	-0.45	(-0.48, -0.41)	(-0.51, -0.36)
j_0	-0.35	0.01	0.01	(-0.11, 0.14)	(-0.33, 0.35)
s_0	-1.3	0.4	0.4	(-0.3, 1.0)	(-1.2, 1.8)
l_0	14.7	-0.6	-1.0	(-4.6, 3.7)	(-11.3, 11.7)

TABLE I: Constraints on the cosmographic parameters by jointly fitting the Union2 SNeIa sample and the BAO data. Columns are as follows: 1. parameter id; 2. best fit; 3., 4. mean and median from the marginalized likelihood; 5., 6. 68 and 95% confidence ranges.

to track the ΛCDM expression with an error less than 1% over the full redshift range. We have checked that this is the case also for the angular diameter distance $D_A = D_L(z)/(1+z)^2$ and the Hubble parameter $H(z)$ which, however, we expand only up to the fourth order to avoid introducing a further cosmographic parameter.

In order to constrain the parameters (h, q_0, j_0, s_0, l_0), we use both the Union2 SNeIa dataset [28] and the BAO data from the analysis of the SDSS seventh release [29]. We then consider the following likelihood function:

$$\mathcal{L}(p) = \mathcal{L}_{SNeIa}(\mathbf{p}) \times \mathcal{L}_{BAO}(\mathbf{p}) \tag{5.1}$$

where $\mathbf{p}$ is the set of model parameters and we have defined the likelihood function for the probe i as:

$$\mathcal{L}_i(\mathbf{p}) = \frac{1}{(2\pi)^{\mathcal{N}_i/2}|\mathbf{C}_i|^{1/2}} \exp\left(-\frac{\mathbf{\Delta}_i^T \mathbf{C}_i^{-1} \mathbf{\Delta}_i}{2}\right) . \tag{5.2}$$

For SNeIa, $\mathbf{\Delta}_{SNeIa}$ is $\mathcal{N}_{SNeIa}$ (with $\mathcal{N}_{SNeIa} = 557$) column vector with elements computed as:

$$\mathbf{\Delta}_{SNeIa,j} = \mu_{obs}(z_j) - \mu_{th}(z_j, \mathbf{p}) \tag{5.3}$$

$$\mu_{th}(z) = 25 + 5 \log D_L(z, \mathbf{p}) , \tag{5.4}$$

while the $\mathbf{C}_{SNeIa}$ is a diagonal matrix. For BAO, we set:

$$\mathbf{\Delta}_{BAO,j} = d_{obs}(z) - d_{th}(z_j, \mathbf{p}) \tag{5.5}$$

$$d_{th}(z, \mathbf{p}) = \frac{r_s(z_d)}{D_V(z, \mathbf{p})} = r_s(z_d) \left[\frac{(1+z)^2 D_A^2(z, \mathbf{p}) cz}{H(z, \mathbf{p})}\right]^{-1/3} , \tag{5.6}$$

where we set the sound horizon distance to the drag redshift as $r_s(z_d) = 152.6$ Mpc. Percival et al. [29] provide estimates of d_z for $z = (0.20, 0.35)$ and the corresponding covariance matrix that we use as input in Eq.(5.5). We remember the reader that we use a fifth order expansion in z for both $D_L(z)$ and $D_A(z)$, while $H(z)$ is expanded to the fourth order only. Since the BAO data are at low redshift, the resulting approximated expression for $d_{th}(z)$ closely follows the exact values. Finally, we also use a Gaussian prior on h from local distance measurement so that (5.2) reduces to a Gaussian centred on $h = 0.742$ and with variance $\sigma_h = 0.036$ [30].

In order to sample the five dimensional parameter space, we use a Markov Chain Monte Carlo algorithm running two chains (with 125000 point each) and checking the convergence according to the Gelman - Rubin criterium ($R - 1 < 0.1$). The resulting constraints are summarized in Table I where we give the best fit parameters and the constraints over the single p_i obtained by marginalizing over the other ones. As a general remark, we find that these constraints are in agreement with previous constraints in literature [23, 31]. Note, however, that our confidence ranges turn out to be narrower than usually found. This is likely due to our inclusion of the lerk parameter l_0. In a sense, we are now better approximating the (unknown) actual distances and Hubble parameter so that not all the possible combinations of (h, q_0, j_0, s_0) are possible, but only the ones that are compatible with the constrained l_0.

In order to translate our constraints on the cosmographic parameters on similar constraints on $f(R)$ and its derivatives, we should just use Eqs.(3.22) - (3.32) evaluating them along the final coadded and thinned chain and then looking at the corresponding histograms. To this end, however, we should set also the values of Ω_M and Ω_{r_c} (and hence $r_c H_0 = 1/2\sqrt{\Omega_{r_c}}$) which are not constrained by the fitting analysis described before. To partially overcome this difficulty, we adopt the following strategy. Defining for shortness

$$f_0 = \frac{f(R_0)}{6H_0^2} \quad , \quad f_2 = \frac{f''(R_0)}{(6H_0^2)^{-1}} \quad , \quad f_3 = \frac{f'''(R_0)}{(6H_0^2)^{-2}} \quad ,$$

we first constrain these quantities setting $\Omega_{r_c} = 10^{-4}$ and varying Ω_M along the chain using $\Omega_M = \omega_M h^{-2}$ with the physical matter density $\omega_M = 0.1329$ in agreement with the WMAP7 data. Note that we are neglecting the uncertainty on ω_M since it is much lower than those on the cosmographic parameters. We also stress that, although the fiducial value for ω_M has been obtained for a ΛCDM model, it should be unchanged for any model which reduces to the GR + matter domination at the CMBR epoch as is our case. We can then scale the results to a different value of $r_c H_0$ noting that, by simple algebra, we get from Eq.(3.22):

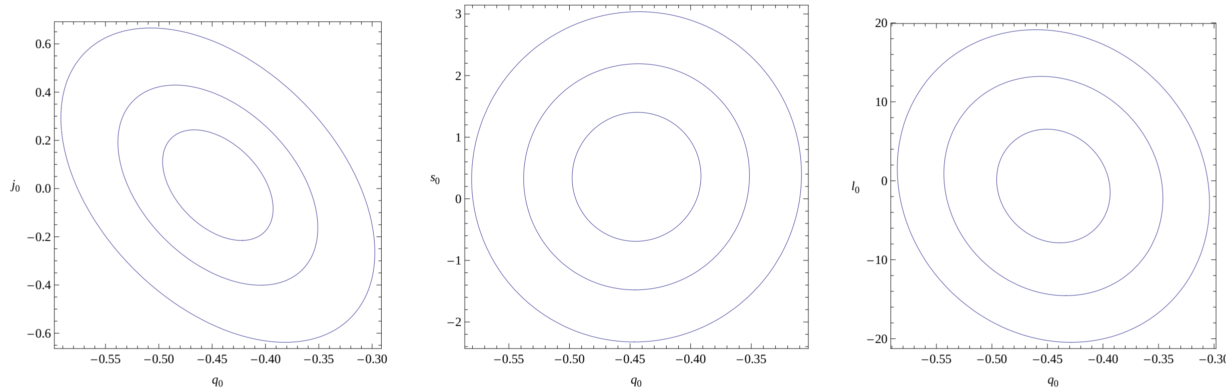

FIG. 1: Isolikelihood (68, 95 and 99% CL) contours for the fit to the SNeIa and BAO data. In each panel, we marginalize the given cosmographic parameter with respect to q_0.

x	x_{BF}	$\langle x \rangle$	x_{med}	68% CL	95% CL
f_0	0.897	0.912	0.912	(0.876, 0.949)	(0.828, 0.992)
f_2	0.126	0.163	0.161	(0.140, 0.185)	(0.116, 0.220)
f_3	-0.130	-0.139	-0.142	(-0.181, -0.101)	(-0.240, -0.004)
α_0	-0.0190	-0.0168	-0.0167	(-0.0180, -0.0155)	(-0.0201, -0.0141)
β_0	1.0190	1.0168	1.0167	(1.0155, 1.0180)	(1.0141, 1.0201)
α_2	0.0130	0.0190	0.0191	(0.0172, 0.0209)	(0.0137, 0.0236)
β_2	0.9870	0.9810	0.9809	(0.9791, 0.9828)	(0.9764, 0.9863)
α_3	3.1091	0.0071	0.0100	(0.0037, 0.0140)	(-0.0272, 0.0263)
β_3	-2.1090	0.9929	0.9899	(0.9860, 0.9962)	(0.9736, 1.0272)

TABLE II: Constraints on the fiducial f_i values and on the scaling coefficients (α_i, β_i) from the Markov Chain for the cosmographic parameters. Columns are as in Table I.

$$\frac{f_i}{f_i^{fid}} = \alpha_i \frac{(r_c H_0)_{fid}}{r_c H_0} + \beta_i = \alpha_i \left(\frac{\Omega_{r_c}^{fid}}{\Omega_{r_c}} \right)^{1/2} + \beta_i \quad (5.7)$$

with the quantities labelled fid are obtained for the fiducial Ω_{r_c} value and we have defined (for $i = 0, 2, 3$):

$$\alpha_i = \frac{\mathcal{C}_i}{(\mathcal{A}_i \Omega_M + \mathcal{B}_i)(r_c H_0)_{fid} + \mathcal{C}_i} \quad (5.8)$$

$$\beta_i = \frac{(\mathcal{A}_i \Omega_M + \mathcal{B}_i)(r_c H_0)_{fid}}{(\mathcal{A}_i \Omega_M + \mathcal{B}_i)(r_c H_0)_{fid} + \mathcal{C}_i} \ . \quad (5.9)$$

The constraints on the fiducial f_i and the scaling parameters (α_i, β_i) obtained by evaluating these quantities along the Markov chain for the cosmographic parameters are summarized in Table II. Considering the median values and the quite narrow confidence ranges, we find that that the scaling parameters (α_i, β_i) are well consistent with the f_i being linear functions of the inverse of the crossover scale r_c hence allowing us to easily estimate the impact of uncertainties on this parameter on the final estimate of the present day values of $f(R)$ and its derivatives. Somewhat surprisingly, the fiducial f_i are reasonably well constrained notwithstanding the large uncertainties on the cosmographic parameters. Such a result can be qualitatively understood noting that f_i depend on (q_0, j_0, s_0, l_0) through a ratio of coefficients so that it is possible that a variation in the numerator is compensated by a similar variation in the denominator in such a way that the final f_i is unaltered. As a consequence, the dependence on the cosmographic parameters is made weaker thus reducing the impact of the parameters uncertainties.

VI. CONCLUSIONS

Cosmography is a useful method to give a picture of the observed universe considering minimal assumptions (isotropy, homogeneity, Taylor series expansion of distances) without choosing any dynamical model a priori.

In this paper, we have taken into account the problem to test brane-cosmology, where an $f(R)$-term is present in the boundary 4D-action, by cosmography. Being ΛCDM a realistic picture of the today observed universe, we have adopted ΛCDM observational results as priors for our approach. We assumed the $f(R)$ function to be analytical

in order to evaluate the higher-order curvature contributions with respect to general relativity contribution, i.e. $f(R) = R$. The results are encouraging since small higher-order deviations with respect to general relativity give dynamical behaviors, consistent with observed cosmic acceleration, *without* introducing dark energy terms.

However, the approach should be consistently probed at small, medium and high redshift by selecting suitable standard candles or, at least, reliable distance indicators at any scale. Despite of this technical difficulty, the method outlined here deserves further investigations since it is connecting a fundamental theory, as the DGP-brane model, with data coming from precision cosmology. We have here addressed this point in a preliminary way by only using SNeIa and BAO, but other probes (such as GRBs) may be added to further narrow the constraints on the present day values of $f(R)$ and its second and third derivatives with respect to R.

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